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DUALITY AND RESTORATION OF MANIFEST SUPERSYMMETRY

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ABSTRACT

World-sheet and spacetime supersymmetries that are manifest in some string backgrounds may not be so in their T-duals. Nevertheless, they always remain symmetries of the underlying conformal field theory. In previous work the mechanism by which T-duality destroys manifest supersymmetry and gives rise to non-local realizations was found. We give the general conditions for a 2-dim $N = 1$ supersymmetric σ -model to have non-local and hence non-manifest extended supersymmetry. We then examine T-duality as a mechanism of restoring manifest supersymmetry. This happens whenever appropriate combinations of non-local parafermions of the underlying conformal field theory become local due to non-trivial world-sheet effects. We present, in detail, an example arising from the model $SU(2)/U(1) \otimes SL(2, \mathbb{R})/U(1)$ and obtain a new exact 4-dim axionic instanton, that generalizes the $SU(2) \otimes U(1)$ semi-wormhole, and has manifest spacetime as well as $N = 4$ world-sheet supersymmetry. In addition, general necessary conditions for abelian T-duality to preserve manifest $N = 4$ world-sheet supersymmetry are derived and applied to WZW models based on quaternionic groups. We also prove some theorems for σ -models with non-local $N = 4$ world-sheet supersymmetry.

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1. Introduction

T-duality is a stringy property that provides an equivalence between strings propagating in different backgrounds [1,2]. Exploration of the interplay between T-duality and other symmetries sheds light on the role and the validity of the effective field theories describing them. The most interesting such interplay is the one with supersymmetry. It was noticed in some examples that abelian T-duality leads to apparent violations of extended $N = 4$ world-sheet supersymmetry as well as of spacetime supersymmetry [3] (Additional examples of that kind were soon provided [4]). However, if T-duality indeed provides an equivalence between strings in different backgrounds, in accordance with general ideas developed by string theorists over the past years, then it should not lead to a real breaking of other genuine symmetries such as supersymmetry.

What has been just stated is nothing but a paradox and has a natural explanation. That is, in certain cases, non-local world-sheet effects associated with the T-duality transformation replace a local realization of supersymmetry with a non-local one [5]. It is at the conformal field theory (CFT) level where one sees the equivalence of the various descriptions and realizations [5]. This point of view was also advocated in [6,7]. In fact non-local realizations are something quite common and natural in (super)CFT. For instance, realizations of the $N = 4$ superconformal algebra using parafermions which are non-local objects in the sense that they have non-local operator product expansions (OPEs) [8]. To make the antithesis it should be pointed out that all theorems that were proved, for instance in the context of 2-dim supersymmetric σ -models [9-17], assumed local realizations of supersymmetry, i.e. underlying complex structures that are local functions of the target space variables. The reason that in some cases supersymmetry seems to be destroyed by duality is that a non-locally realized supersymmetry cannot be distinguished from a lost one if anyone of these theorems is used as a criterion. It is one of the main purposes of the present paper to further bridge the gap between the CFT and σ -model approaches by deriving conditions for non-local realizations of world-sheet supersymmetry to exist and revising some of these theorems.

There is an additional motivation from a string phenomenological point of view. If duality breaks or, as we shall see, restores manifest supersymmetry¹ then this phenomenon should be incorporated in supersymmetry breaking scenarios considered in a string phenomenological context. To that effect it is very interesting to uncover, and when possible relate, all mechanisms that lead away from or to a manifest realization of supersymmetry.

This paper is organized as follows: In section 2 we start from a general 2-dim σ -model with $N = 1$ world-sheet supersymmetry and we find the conditions for being able to have non-locally realized extended supersymmetry. An exploration of the consequences is done in appendix A in order not to interrupt the flow of the paper.

In section 3 we develop a criterion for when it is possible to obtain a background with manifest $N = 4$ extended supersymmetry (and target space supersymmetry) via a duality transformation. Its practical use is based on the fact that only knowledge of one (instead of three) complex structure is required. We apply this to several cases with notable example being WZW models corresponding to quaternionic manifolds. We prove that any marginal deformation of them in the Cartan torus (equivalent to an $O(d, d, \mathbb{R})$ transformation) necessarily breaks their manifest $N = 4$ world-sheet supersymmetry.

In section 4 we consider in detail a class of models arising from duality transformation on the background for $SU(2)/U(1) \otimes SL(2, \mathbb{R})/U(1)$. We show that when a combination of parafermions of the CFT becomes local a manifest $N = 4$ world-sheet and spacetime supersymmetry emerges. This is the prototype example of what we will call duality restoration of manifest supersymmetry. The corresponding model is a new axionic instanton and its spacetime interpretation is that of a generalized semi-wormhole. This model was also considered in [7] in relation to what was called dynamical restoration of manifest spacetime supersymmetry.

We end the paper in section 5 with concluding remarks and comments on directions for future work.

¹ The term manifest supersymmetry is equivalent to the term locally realized supersymmetry in this paper.

We have also written two appendices. In appendix A we prove some general theorems for backgrounds having non-locally realized $N = 4$ world-sheet supersymmetry and point out the differences from the cases where supersymmetry is realized locally. In appendix B we prove in a class of models that, making the moduli parameter dynamical (namely, coordinate dependent) is equivalent to performing specific duality transformations. This shows that, at least in these cases, the two mechanism of restoring manifest supersymmetry are equivalent.

2. General conditions for non-local realizations of supersymmetry

The action of a 2-dim σ -model with $N = 1$ supersymmetry is given by [11,13,15]

$$\begin{aligned} S(x, \Psi_+, \Psi_-) = & \frac{1}{2} \int Q_{\mu\nu}^+ \partial_+ x^\mu \partial_- x^\nu + i G_{\mu\nu} \Psi_+^\mu (\partial_- \Psi_+^\nu + (\Omega^+)^{\nu}_{\lambda\rho} \partial_- x^\lambda \Psi_+^\rho) \\ & + i G_{\mu\nu} \Psi_-^\mu (\partial_+ \Psi_-^\nu + (\Omega^-)^{\nu}_{\lambda\rho} \partial_+ x^\lambda \Psi_-^\rho) + \frac{1}{2} R_{\mu\nu\rho\lambda}^- \Psi_+^\mu \Psi_+^\nu \Psi_-^\rho \Psi_-^\lambda, \end{aligned} \quad (2.1)$$

where $G_{\mu\nu}$ and $B_{\mu\nu}$ are the metric and the antisymmetric tensor and $Q_{\mu\nu}^\pm \equiv G_{\mu\nu} \pm B_{\mu\nu}$. The generalized connections are defined including the torsion $H_{\mu\nu\rho} \equiv \partial_{[\rho} B_{\mu\nu]}$, i.e., $(\Omega^\pm)^{\rho}_{\mu\nu} = \Gamma_{\mu\nu}^\rho \pm \frac{1}{2} H_{\mu\nu}^\rho$, and $R_{\mu\nu\rho\lambda}^\pm = R_{\rho\lambda\mu\nu}^\mp$ are the corresponding curvature tensors. Notably, any background can be made $N = 1$ supersymmetric. In contrast, it is well known that extended $N = 2$ supersymmetry [9,10,12] requires that the background is such that an (almost) complex (hermitian) structure $(F^\pm)^\mu{}_\nu$, for each chiral sector, exists. The conditions to be satisfied are

$$(F^\pm)^\mu{}_\lambda (F^\pm)^\lambda{}_\nu = -\delta^\mu{}_\nu, \quad F_{\mu\nu}^\pm + F_{\nu\mu}^\pm = 0, \quad D_\mu^\pm (F^\pm)^\lambda{}_\rho = 0, \quad (2.2)$$

where $F_{\mu\nu}^\pm \equiv G_{\mu\lambda} (F^\pm)^\lambda{}_\nu$ and the generalized connections are used to define the covariant derivatives. The first two conditions guarantee that the commutator of two new supersymmetries gives the same translation as that of two old ones and that no translation is generated by commuting an old and a new supersymmetry. The third condition requires that the complex structures are covariantly constant and guarantee the invariance

of the quadratic in the fermions terms. Moreover, its integrability condition assures the invariance of the quartic in the fermions terms. Then one can check that the following invariances of the action hold

$$S(x, F^+ \Psi_+, \Psi_-) = S(x, \Psi_+, F^- \Psi_-) = S(x, \Psi_+, \Psi_-) . \quad (2.3)$$

Similarly, $N = 4$ extended supersymmetry [10,12,15] requires that, for each sector, there exist three complex structures $(F_I^\pm)^\mu{}_\nu$, $I = 1, 2, 3$. Each one of them satisfies separately (2.2), and in addition they obey

$$F_I^\pm F_J^\pm = -\delta^{IJ} + \epsilon^{IJK} F_K^\pm . \quad (2.4)$$

The previous results were derived under the crucial assumption that the complex structures are local functions of the target space variables. This was rather unquestionable in the past, but recent work [5,6] shows that non-local complex structures are equally acceptable in a string theoretical setting and are directly related [5] to parafermions of the underlying (super)CFT corresponding to the σ -model (2.1). Therefore it is interesting to investigate the conditions under which (2.1) has non-locally realized extended supersymmetry, in the sense that the corresponding complex structures are allowed to depend non-locally on the bosonic coordinates and on the world-sheet fermions. Namely, let $F^\pm = F^\pm(\vec{\theta}, x^\mu)$, where $\vec{\theta}$ is a vectorial notation for N functionals. A general ansatz for them, consistent with scaling arguments, is

$$\vec{\theta} \equiv \int (\vec{C}_\mu^+ \partial_+ x^\mu + i \vec{C}_{\mu\nu}^+ \Psi_+^\mu \Psi_+^\nu) d\sigma^+ + (\vec{C}_\mu^- \partial_- x^\mu + i \vec{C}_{\mu\nu}^- \Psi_-^\mu \Psi_-^\nu) d\sigma^- , \quad (2.5)$$

where the tensors \vec{C}_μ^\pm and $\vec{C}_{\mu\nu}^\pm$ depend locally on the x^μ 's. These, as well as the complex structures themselves, will be determined by requiring that the action (2.1) has still the invariances (2.3). In working out the details first we examine the vanishing of anomaly terms that are quadratic in the fermions. This and the requirement that the commutator

of an old and a new supersymmetry does not generate a translation give the first two conditions in (2.2) and

$$D_\mu^\pm F_{\alpha\beta}^\pm + (\vec{C}_\mu^\mp \cdot \partial_{\vec{\theta}}) F_{\alpha\beta}^\pm = 0 , \quad (2.6)$$

where by definition ordinary covariant derivatives do not act on the integrand of $\vec{\theta}$. Its contribution has been taken into account by the second term in (2.6). The latter is the analog of the third condition in (2.2). We see that the complex structures are no longer covariantly constant in the ordinary sense.² It remains to examine the invariance of the quartic in the fermions terms in (2.1) under (2.3). In the case of local complex structures this term is invariant by itself thanks to the integrability condition of the third equation in (2.2). However, in our case we get extra contributions from the variation of the quadratic in the fermions terms in the action. The condition that the combined result is zero can be written as

$$R_{\mu\nu\alpha}^\pm \gamma F_{\gamma\beta}^\pm - R_{\mu\nu\beta}^\pm \gamma F_{\gamma\alpha}^\pm - 2\vec{C}_{\mu\nu}^\mp \cdot \partial_{\vec{\theta}} F_{\alpha\beta}^\pm = 0 . \quad (2.8)$$

Next we consider the integrability condition of (2.6) which is

$$\begin{aligned} & [D_\mu^\pm, D_\nu^\pm] F_{\alpha\beta}^\pm + D_{[\mu}^\pm \vec{C}_{\nu]}^\mp \cdot \partial_{\vec{\theta}} F_{\alpha\beta}^\pm + \vec{C}_{[\nu}^\mp \cdot \partial_{\vec{\theta}} D_{\mu]}^\pm F_{\alpha\beta}^\pm \\ &= R_{\mu\nu\alpha}^\pm \gamma F_{\gamma\beta}^\pm - R_{\mu\nu\beta}^\pm \gamma F_{\gamma\alpha}^\pm \mp H_{\mu\nu}^\gamma D_\gamma^\pm F_{\alpha\beta}^\pm \\ &+ (\partial_{[\mu} \vec{C}_{\nu]}^\mp \mp H_{\mu\nu}^\gamma \vec{C}_\gamma^\mp) \cdot \partial_{\vec{\theta}} F_{\alpha\beta}^\pm + \vec{C}_{[\nu}^\mp \cdot \partial_{\vec{\theta}} D_{\mu]}^\pm F_{\alpha\beta}^\pm = 0 , \end{aligned} \quad (2.9)$$

where the second line arises from the first term of the first line and is the usual result one obtains from the commutator of two covariant derivatives. The first term of the third

² The non-local complex structures are covariantly constant with respect to covariant derivatives which have in their definitions ordinary derivatives ∂_μ replaced by $\partial_\mu + \vec{C}_\mu^\mp \cdot \partial_{\vec{\theta}}$. It should also be noted that ansatz (2.5) excludes the possibility of path ordered (when off shell) Wilson lines. The most general, ansatz-free, conditions on the complex structures are

$$D_\mu^\pm F_{\alpha\beta}^\pm \partial_\mp x^\mu + \tilde{\partial}_\mp F_{\alpha\beta}^\pm = 0 , \quad (2.7)$$

where the tilded world-sheet derivative acts only on the non-local part of the complex structure and all dependence on world-sheet fermions is ignored. Obviously if we make the ansatz (2.5), conditions (2.6) are recovered.

line is an explicit rewriting of the second term of the first line after taking into account that only the antisymmetric part of the connections contributes. Notice that the chain rule and all the usual properties of the covariant derivatives are valid since, as already mentioned, by definition they do not act on the integrand of $\vec{\theta}$. Now using (2.6) to rewrite the covariant derivatives we see that the two terms proportional to the torsion cancel each other and that the last term is zero. The final result is

$$R_{\mu\nu\alpha}^{\pm}{}^{\gamma}F_{\gamma\beta}^{\pm} - R_{\mu\nu\beta}^{\pm}{}^{\gamma}F_{\gamma\alpha}^{\pm} + (\partial_{\mu}\vec{C}_{\nu}^{\mp} - \partial_{\nu}\vec{C}_{\mu}^{\mp}) \cdot \partial_{\vec{\theta}}F_{\alpha\beta}^{\pm} = 0. \quad (2.10)$$

A non-vanishing third term causes the complex structures not to commute with the generators of the holonomy group $(M_{\mu\nu}^{\pm})_{\alpha}^{\beta} = R_{\mu\nu\alpha}^{\pm}{}^{\beta}$. Comparison of (2.8) with (2.10) determines that

$$\vec{C}_{\mu\nu}^{\pm} = -\frac{1}{2}(\partial_{\mu}\vec{C}_{\nu}^{\pm} - \partial_{\nu}\vec{C}_{\mu}^{\pm}). \quad (2.11)$$

Notice that if $\vec{C}_{\mu}^{\pm} = \partial_{\mu}\vec{\Lambda}(x)$, then $\vec{C}_{\mu\nu}^{\pm} = 0$ and $\vec{\theta} = \vec{\Lambda}$. Thus in this case, the complex structure is local and the supersymmetry manifest. This observation will be the key to understand restoration mechanisms of manifest supersymmetry, as we will soon discuss.

In the case that the non-localities arise from a duality transformation with respect to a Killing vector $\partial/\partial x^0$ one obtains, $C_{\mu}^{\pm} = \pm Q_{\mu 0}^{\pm}$ (defined in the dual model of course) and θ corresponds to the Killing coordinate of the original model that is non-locally related to the variables of its dual [5]. Also the non-local complex structure should automatically satisfy (2.6) by construction. This was proved in [6], where the question, what kind of equation the non-local complex structures arising from the duality transformation obey, led to the analog of (2.6). Similarly, we find that for duality with respect to N commuting isometries, the components of the vector \vec{C}_{μ}^{\pm} are $(C^{\pm})_{\mu}^a = \pm Q_{\mu a}^{\pm}$, $a = 1, 2, \dots, N$.

Notice that derivatives of F^{\pm} with respect to components of $\vec{\theta}$ are also candidates for new independent complex structures. For simplicity let us consider the case of one functional θ . It is clear from (2.6) that $\partial_{\theta}F^{\pm}$ satisfies the same equation and some algebra shows that so does $F^{\pm}\partial_{\theta}F^{\pm}$. It is easy to see that $[(\partial_{\theta}F^{\pm})^2, F^{\pm}] = 0$ and therefore after

a suitable normalization $\partial_\theta F^\pm$ are indeed complex structures and so are $F^\pm \partial_\theta F^\pm$. This observation will be further exploited in appendix A.

It should be clear that from our point of view any mechanism that can transform $\vec{\theta}$ into a local function would simultaneously restore manifest supersymmetry. One possibility would be to take advantage of possible isometry groups that the background might have and perform a T-duality transformation. Since that will transform the world-sheet derivatives and the fermions (we know explicitly how in the case of abelian duality [18,5,6] and in principle we can determine that for the Principal Chiral Models using [19]) in the integrand of $\vec{\theta}$ there is a chance, depending on the specific background, that the integrand becomes a total derivative on the world-sheet. Then as it was discussed below (2.11) the complex structures in the dual theory would become local and the supersymmetry would be manifest. This will be demonstrated in detail with an example in section 4. Another mechanism could be what has been called dynamical restoration of manifest supersymmetry [7]. According to the general philosophy this corresponds to making certain moduli parameters coordinate-dependent [20,21] (for earlier work see [22]) and then demanding that this drastic modification still preserves conformal invariance. It might happen that manifest supersymmetry is also restored in the process ([7] and subsection 4.1 of the present paper). Although we have no general proof, we believe that this mechanism is always equivalent to a restoration via particular duality transformations. We prove this claim for a class of one parameter moduli models, a particular example of which is the one considered in section 4. For this reason we did not focus on the dynamical restoration mechanism in this paper.

3. Conditions for manifest supersymmetry under duality

So far we have not required that the σ -model action (2.1) had any special isometries. Let us consider the case where there is one (at least) Killing symmetry corresponding to the Killing vector $\partial/\partial x^0$ in the adapted coordinate system, in which the background fields do not depend on x^0 . This need not be the case for other geometrical objects of interest,

such as complex structures [5]. Let us assume that the model (2.1) has extended $N = 2$ (at least) supersymmetry, which in the adapted coordinate system is always manifest, and that we have determined the corresponding complex structure in each chiral sector. We would like to derive a necessary condition need to be satisfied for the dual model to actually have manifest $N = 4$ extended supersymmetry. In practical situations such a check will be useful since it does not require knowledge of any additional complex structures. A related problem will be to find conditions that need to be satisfied in order that both (2.1) and its dual have manifest $N = 4$ extended supersymmetry.³ Let us recall that the assumption that there exist three (local) complex structures satisfying (2.4) leads to the strong conditions [15]

$$\tilde{R}_{\mu\nu\alpha\beta}^{\pm}(\tilde{F}_I^{\pm})^{\alpha\beta} = 0, \quad I = 1, 2, 3, \quad (3.1)$$

where we have written them with tildes having in mind that they should be satisfied in the dual to (2.1) model. We would like to express these conditions in terms of tensors defined in the original model (2.1). The complex structures in the dual model are [24,25,5]

$$(\tilde{F}_I^{\pm})_{\mu\nu} = (A_{\pm}^T F_I^{\pm} A_{\pm})_{\mu\nu}, \quad (3.2)$$

where

$$(A_{\pm})^{\mu}_{\nu} = \begin{matrix} & 0 & j \\ 0 & \left(\begin{array}{cc} \pm G_{00}^{-1} & -G_{00}^{-1} Q_{j0}^{\pm} \\ 0 & \delta^i_j \end{array} \right) & \\ i & & \end{matrix}. \quad (3.3)$$

The curvature tensors of the dual model can be extracted from an expression in [6]. We find

$$\tilde{R}_{\mu\nu\alpha\beta}^{\pm} = (A_{\mp})^{\lambda}_{\mu} (A_{\mp})^{\rho}_{\nu} (A_{\pm})^{\gamma}_{\alpha} (A_{\pm})^{\delta}_{\beta} \left(R_{\lambda\rho\gamma\delta}^{\pm} + \frac{1}{2} G_{00}^{-1} \partial_{[\lambda} Q_{0\rho]}^{\pm} \partial_{[\gamma} Q_{0\delta]}^{\mp} \right). \quad (3.4)$$

Combining (3.2) with (3.4) we see that (3.1) takes the form

$$R_{\mu\nu\alpha\beta}^{\pm} (F_I^{\pm})^{\alpha\beta} + G_{00}^{-1} \partial_{[\mu} Q_{0\nu]}^{\pm} \partial_{\alpha} Q_{0\beta}^{\mp} (F_I^{\pm})^{\alpha\beta} = 0, \quad I = 1, 2, 3. \quad (3.5)$$

³ By means of the relation between target space and extended world-sheet supersymmetry [23] these will also be necessary conditions for having manifest target space supersymmetry.

If the original model had manifest $N = 4$ then the first term is zero and the condition for having manifest $N = 4$ in the dual model reduces to the vanishing of the second term in (3.5). Let us emphasize that (3.5) is only a necessary condition for $N = 4$ supersymmetry and only its violation leads to a definite conclusion that the duality transformation has broken manifest $N = 4$. Nevertheless, we know of no examples where the reverse is not also true, i.e. satisfying (3.5) seems to always lead to a manifestly $N = 4$ supersymmetric dual model.

As examples let us briefly consider 4-dim pure gravitational backgrounds with $N = 4$ extended supersymmetry, which are known to be hyper-kähler self-dual manifolds, that in addition have one Killing symmetry. A complete classification of them exists and depends on whether or not the covariant derivative of the corresponding Killing vector is self-dual [26]. Accordingly, the Killing vector is of the translational or the rotational type. In the translational case the general forms of the metric and the three complex structures have been found in [27] and in [28] respectively, whereas in the rotational case in [26] and [5]. One can explicitly check that in the translational case $\partial_\alpha G_{0\beta} F_I^{\alpha\beta}$ is 0, for all three complex structures. In the rotational case, we have checked the same expression for the complex structure adapted to the Killing vector⁴ and we found that it is 2.⁵ Since $\partial_{[\mu} G_{0\nu]} \neq 0$ we conclude that in general a duality transformation with respect to a rotational Killing vector cannot preserve $N = 4$ as a locally realized supersymmetry. This is in full agreement with the conclusion reached in [5] on the basis that two of the complex structures in the rotational case become non-local under duality.

Next we consider marginal deformations of WZW models based on quaternionic groups (this implies, a dimensionality that is a multiple of four, three complex structures and manifest $N = 4$; for a complete analysis see [29]) by current-bilinears in the Cartan torus.

⁴ A complex structure adapted to a Killing vector is by definition one that is a singlet under transformations generated by this Killing vector.

⁵ For the other two complex structures that form a doublet this expression is zero. A general explanation of this is given in appendix A.

This is equivalent to duality transformations [30] (strictly speaking $O(d, d, \mathbb{R})$ transformations). We will prove using (3.5) that any such deformation leads to a breaking of manifest $N = 4$. Moreover, with no additional effort, we will show that if a WZW model has $N = 2$ only (and a multiple of four dimensionality) no marginal deformation in the Cartan torus can result in a dual model that has manifest $N = 4$. We examine the case of one parameter duality transformations. The proof starts by recalling that for any WZW model the generalized curvature tensors are identically zero [31], i.e. $R_{\mu\nu\rho\lambda}^{\pm} = 0$, reflecting the parallelizability of the corresponding group manifold. Thus the second term in (3.5) should vanish by itself for both the $+$ and the $-$ sign. One way this can be satisfied is by having $G_{00} = \text{const.}$ and $Q_{0i}^{-} = 0$, i.e., the duality corresponds to a chiral isometry. However, this is not an interesting case since then the background is self-dual [32,18,6]. Thus for a duality with respect to any other kind of isometry the only possibility is to have $\partial_{\alpha} Q_{0\beta}^{\mp} (F_I^{\pm})^{\alpha\beta} = 0$ for both signs and for all three complex structures. The simplest type of isometry corresponding to a non-trivial duality transformation is a mixed one of chiral and anti-chiral type. Specifically we parametrize the group element $g \in G$ as

$$g = e^{i\theta_L T} h(x) e^{i\theta_R T} , \quad \theta_L = (Q + 1)\tau + \psi , \quad \theta_R = (1 - Q)\tau - \psi , \quad (3.6)$$

where T is a Cartan generator and Q is a constant modulus. Then the corresponding WZW action [33] $S = k I_{wzw}(g)$ takes the form

$$\begin{aligned} I_{wzw}(g) = & I_{wzw}(h) + \frac{1}{\pi} \int \left((1 + Q^2) + (1 - Q^2)\Sigma \right) \partial_+ \tau \partial_- \tau + (1 - \Sigma) \partial_+ \psi \partial_- \psi \\ & + \left((Q - 1) - (1 + Q)\Sigma \right) \partial_+ \tau \partial_- \psi + \left((1 + Q) + (1 - Q)\Sigma \right) \partial_+ \psi \partial_- \tau \\ & + \left((1 + Q) \partial_+ \tau + \partial_+ \psi \right) J_i^- \partial_- x^i - J_i^+ \partial_+ x^i \left((Q - 1) \partial_- \tau + \partial_- \psi \right) , \end{aligned} \quad (3.7)$$

where we used the definitions $(\partial_i \equiv \partial/\partial x^i)$

$$J_i^+ = -i \text{Tr}(T h^{-1} \partial_i h) , \quad J_i^- = -i \text{Tr}(T \partial_i h h^{-1}) , \quad \Sigma = \text{Tr}(T h T h^{-1}) . \quad (3.8)$$

The action (3.7) will be dualized with respect to the Killing vector $\partial/\partial \tau$ and the matrix $Q_{\mu\nu}^{\pm}$ can be easily read off. Notice that, if the modulus $Q = \pm 1$ then the isometry becomes

chiral. Assuming that $Q \neq \pm 1$ and using the complex structures F_3^\pm adapted to the Killing vector $\partial/\partial\tau$, the conditions $\partial_\alpha Q_{0\beta}^\mp (F_3^\pm)^{\alpha\beta} = 0$ become

$$\partial_i \Sigma((Q \pm 1)(F_3^\pm)^{\tau i} + (F_3^\pm)^{\psi i}) \mp \partial_i J_j^\pm (F_3^\pm)^{ij} = 0. \quad (3.9)$$

The above equations cannot be satisfied, as we shall show next. Let $T^A = \{T^1, T^a\}$, $a = 2, 3, \dots, \dim(G)$ denote the Lie algebra generators with T^1 the generator in the Cartan subalgebra we have been using, i.e., $T^1 \equiv T$. It is also convenient to define

$$L_\mu^A = -i\text{Tr}(T^A g^{-1} \partial_\mu g), \quad R_\mu^A = -i\text{Tr}(T^A \partial_\mu g g^{-1}), \quad C^{AB} = \text{Tr}(T^A g T^B g^{-1}), \quad (3.10)$$

with (3.8) related to them in an obvious way. Then a straightforward computation gives

$$\begin{aligned} \partial_i J_j^+(F_3^+)^{ij} &= \left(\frac{1}{2}(f^+)^{ab} - L_\tau^a L_i^b (F_3^+)^{\tau i} - L_\psi^a L_i^b (F_3^+)^{\psi i} - L_\tau^a L_\psi^b (F_3^+)^{\tau\psi}\right) f_{ab}^1, \\ \partial_i \Sigma &= -C^{1a} L_i^b f_{ab}^1, \quad L_\tau^a = (1+Q)C^{1a}, \quad L_\psi^a = C^{1a}, \end{aligned} \quad (3.11)$$

where f_{ab}^1 is the relevant Lie algebra structure constant. Note that, $(f^+)^{ab}$ is a Lie algebra complex structure [29], defined as $(F_3^+)_{\mu\nu} = L_\mu^A L_\nu^B (f^+)_{AB}$. Using (3.11) and its analog in the other chiral sector, the conditions (3.9) take the simplified form $f_{ab}^1 (f^\pm)^{ab} = 0$. In the Cartan basis the indices a, b run only over pairs of positive and negative roots $(\alpha, \bar{\alpha})$ of the Lie algebra (if either a or b takes a value in the Cartan subalgebra then the corresponding structure constant vanishes). In this basis $(f^\pm)^\alpha_\alpha = -(f^\pm)^{\bar{\alpha}}_{\bar{\alpha}} = i$ [29] and the conditions become $\sum_\alpha f_{\alpha 1}^\alpha \sim \sum_\alpha \alpha^1 = 0$, where α^1 denotes the component of the positive root in the direction of the Killing vector. Clearly this equality cannot be satisfied.

Thus, we have proved that a one parameter marginal deformation in the Cartan torus of a WZW model for a quaternionic group breaks its manifest $N = 4$ extended supersymmetry. If the original WZW model had only an $N = 2$ we also conclude that duality cannot enlarge the supersymmetry to an $N = 4$. Obviously, the same conclusions are valid for more complicated $O(d, d, \mathbb{R})$ transformations.

Interesting situations arise when none of the terms in (3.5) is zero but their forms are such that the equality is satisfied. A non-vanishing first term implies that only the $N = 2$ supersymmetry was manifest in the original model and that there were two additional ones, non-locally realized. It is only after duality we are actually promoting it to a manifest $N = 4$. This is what we call duality restoration of manifest supersymmetry.

4. An explicit example

In order to demonstrate every aspect related to the general discussion of the previous sections we consider the following background with line element given by

$$ds^2 = d\varphi^2 + \cot^2 \varphi \, dx^2 + d\rho^2 + R^2(\rho) \, dy^2 , \quad (4.1)$$

and zero antisymmetric tensor. It is well known and easy to compute that 1-loop conformal invariance requires that a non-trivial dilaton is induced with

$$\Phi = \ln(\sin^2 \varphi / R'(\rho)) , \quad (4.2)$$

where the function $R(\rho)$ satisfies the differential equation

$$R' = C_1 R^2 + C_2 , \quad (4.3)$$

whose general solution is easy to obtain (see for instance [20]). Depending on the constants C_1, C_2 the solutions correspond to a 4-dim model that can be considered as the tensor product of the coset CFT model $SU(2)/U(1)$ with another $SU(2)/U(1)$ model (if $C_1 C_2 > 0$) or with $SL(2, \mathbb{R})/U(1)$ (if $C_1 C_2 < 0$) or with 2-dim flat space (if $C_1 = 0$) or with the dual to 2-dim flat space (if $C_2 = 0$). The central charge deficit from the classical value $c = 4$ is $\delta c \sim -\alpha'(1 + C_1 C_2)$. Next we introduce the coordinate change

$$x = \psi - \frac{\tau}{2} , \quad y = \psi + \frac{\tau}{2} \quad (4.4)$$

and perform a T-duality transformation with respect to the symmetry generated by the $\partial/\partial\psi$ Killing vector, thus obtaining the background

$$\begin{aligned} d\tilde{s}^2 &= d\rho^2 + d\varphi^2 + \frac{1}{1 + R^2 \tan^2 \varphi} (\tan^2 \varphi \, d\tilde{\psi}^2 + R^2 \, d\tau^2) , \\ \tilde{B}_{\tau\tilde{\psi}} &= \frac{1}{1 + R^2 \tan^2 \varphi} , \quad \tilde{\Phi} = \ln((\cos^2 \varphi + R^2 \sin^2 \varphi)/R') , \end{aligned} \quad (4.5)$$

where we have denoted the dual to ψ variable by $\tilde{\psi}$. Their explicit relation follows in the formulation of abelian T-duality as a canonical transformation [18] (see also [34,19]) and can be found using general formulae that relate world-sheet derivatives [5]

$$\partial_{\pm}\psi = \frac{1}{1 + R^2 \tan^2 \varphi} \left(\pm \tan^2 \varphi \partial_{\pm} \tilde{\psi} + \frac{1}{2} ((1 - R^2 \tan^2 \varphi) \partial_{\pm} \tau) \right) . \quad (4.6)$$

This formula will be very important as we shall shortly see.

We now turn to the question of world-sheet supersymmetry for the background (4.1). There is only one local complex structure that solves (2.2) given by

$$F_3 = \cot \varphi \, d\varphi \wedge dx + R(\rho) \, d\rho \wedge dy , \quad (4.7)$$

for any function $R(\rho)$. Notice that, there is no distinction between the $+$ and the $-$ components since the antisymmetric tensor is zero. However if the locality condition for the complex structures is relaxed one can search for solutions of (2.6) and one finds that there are two for each chiral sector. In order to present them it is first convenient to introduce the parafermionic type, 1-forms

$$\begin{aligned} \Psi_{\pm}^{(1)} &= (d\varphi \pm i \cot \varphi \, dx) e^{\pm i(-x+\theta_1)} , \quad \bar{\Psi}_{\pm}^{(1)} = (d\varphi \mp i \cot \varphi \, dx) e^{\pm i(x+\theta_1)} , \\ \theta_1 &\equiv \int \cot^2 \varphi \partial_+ x d\sigma^+ - \cot^2 \varphi \partial_- x d\sigma^- , \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \Psi_{\pm}^{(2)} &= (d\rho \pm iR \, dy) e^{\pm i(c_2 y + \theta_2)} , \quad \bar{\Psi}_{\pm}^{(2)} = (d\rho \mp iR \, dy) e^{\pm i(-c_2 y + \theta_2)} , \\ \theta_2 &\equiv \int (c_2 - R') \partial_+ y d\sigma^+ - (c_2 - R') \partial_- y d\sigma^- , \end{aligned} \quad (4.9)$$

where c_2 is an arbitrary constant. These have a natural decomposition in terms of $(1,0)$ and $(0,1)$ forms on the string world-sheet

$$\Psi_{\pm}^{(a)} = \Psi_{\pm,+}^{(a)} d\sigma^+ + \Psi_{\pm,-}^{(a)} d\sigma^- , \quad \bar{\Psi}_{\pm}^{(a)} = \bar{\Psi}_{\pm,+}^{(a)} d\sigma^+ + \bar{\Psi}_{\pm,-}^{(a)} d\sigma^- , \quad (4.10)$$

where $a = 1, 2$. It can be easily verified using the classical equations of motion for the model (4.1) that the chiral and anti-chiral conservation laws

$$\partial_- \Psi_{\pm,+}^{(a)} = 0 , \quad \partial_+ \bar{\Psi}_{\pm,-}^{(a)} = 0 , \quad (4.11)$$

are obeyed. In fact in this case $\Psi_{\pm,+}^{(1)}$ and $\bar{\Psi}_{\pm,-}^{(1)}$ are nothing but the classical parafermions for the $SU(2)/U(1)$ coset. However, $\Psi_{\pm,+}^{(2)}$ and $\bar{\Psi}_{\pm,-}^{(2)}$, even though are of the parafermionic type, cannot be identified with classical parafermions of some 2-dim CFT unless the function $R(\rho)$ obeys (4.3), i.e., the corresponding σ -model is conformal. Using the above definitions the expressions for the two non-local complex structures are

$$F_1^+ = \Psi_+^{(1)} \wedge \Psi_+^{(2)} + \Psi_-^{(1)} \wedge \Psi_-^{(2)}, \quad F_2^+ = i\Psi_+^{(1)} \wedge \Psi_+^{(2)} - i\Psi_-^{(1)} \wedge \Psi_-^{(2)}, \quad (4.12)$$

and

$$F_1^- = \bar{\Psi}_+^{(1)} \wedge \bar{\Psi}_+^{(2)} + \bar{\Psi}_-^{(1)} \wedge \bar{\Psi}_-^{(2)}, \quad F_2^- = -i\bar{\Psi}_+^{(1)} \wedge \bar{\Psi}_+^{(2)} + i\bar{\Psi}_-^{(1)} \wedge \bar{\Psi}_-^{(2)}, \quad (4.13)$$

where we have neglected writing explicitly the necessary for non-local complex structures dependence on the world-sheet fermions since it is completely fixed by the bosonic part (see (2.5)(2.11)). Notice that, there is a distinction between the $+$ and the $-$ components even though the antisymmetric tensor is zero. This is a new feature that can only happen in non-local realizations of extended supersymmetry. It is interesting that the local complex structure F_3 , can also be written in terms of the parafermionic 1-forms as

$$F_3 = i\Psi_+^{(1)} \wedge \Psi_-^{(1)} + i\Psi_+^{(2)} \wedge \Psi_-^{(2)} = -i\bar{\Psi}_+^{(1)} \wedge \bar{\Psi}_-^{(1)} - i\bar{\Psi}_+^{(2)} \wedge \bar{\Psi}_-^{(2)}, \quad (4.14)$$

where a simple inspection shows that both alternative expressions reduce to that in (4.7). The structure of the non-locally realized $N = 4$ we have just exhibited in this example is in full agreement with general conclusions in appendix A.

We now turn to the question of world-sheet supersymmetry for the dual background (4.5). Instead of solving the corresponding conditions (2.6) we apply the general formula (3.2) in our case (first we pass to the coordinate system (4.4)) and in addition for the non-local complex structures F_1^\pm and F_2^\pm we use the transformation rules for the world-sheet derivatives (4.6) in order to deduce the transformation of the functionals θ_1, θ_2 defined in (4.8)(4.9). The dual of the local complex structure F_3 , in each chiral sector, is

$$\tilde{F}_3^\pm = \frac{1}{1 + R^2 \tan^2 \varphi} (R d\rho \wedge (d\tau \pm \tan^2 \varphi d\tilde{\psi}) + \tan \varphi (R^2 d\tau \mp d\tilde{\psi}) \wedge d\varphi). \quad (4.15)$$

Indeed it can be explicitly verified that this is a complex structure for the dual model for any function $R(\rho)$. Next we consider the transformation of the non-local complex structures (4.12)(4.13). From (4.8)(4.9) we see that generically the non-localities will persist in the corresponding complex structures of the dual model which will also have non-locally realized $N = 4$ supersymmetry. However, a closer look reveals that there is a particular case where they completely cancel out after the duality is performed. Let us consider the phase factors in $F_{1,2}^\pm$ where all non-localities lie,

$$\begin{aligned} \theta_1 + \theta_2 \pm (c_2 y - x) &= \pm(c_2 - 1)\psi \pm \frac{1}{2}(c_2 + 1)\tau \\ &+ \int ((c_2 - R' + \cot^2 \varphi)\partial_+ \psi + \frac{1}{2}(c_2 - R' - \cot^2 \varphi)\partial_+ \tau) d\sigma^+ - (+ \rightarrow -) , \end{aligned} \quad (4.16)$$

where we have passed to the coordinate system (4.4). Under duality with respect to ψ the world-sheet derivatives $\partial_\pm \psi$ will transform as in (4.6). It is easily now seen that only if we choose $c_2 = 1$ and the function $R(\rho)$ to satisfy

$$R' = 1 - R^2 \quad \Rightarrow \quad R = \tanh \rho \quad \text{or} \quad \coth \rho , \quad (4.17)$$

all non-localities in the dual complex structures disappear.⁶ Indeed then the phase factors (4.16) transform under duality to just $\tilde{\psi} \pm \tau$. The corresponding local complex structures dual to (4.12)(4.13) are then

$$\begin{pmatrix} \tilde{F}_1^\pm \\ \tilde{F}_2^\pm \end{pmatrix} = \begin{pmatrix} \cos(\tau \pm \tilde{\psi}) & \sin(\tau \pm \tilde{\psi}) \\ -\sin(\tau \pm \tilde{\psi}) & \cos(\tau \pm \tilde{\psi}) \end{pmatrix} \begin{pmatrix} f_1^\pm \\ f_2^\pm \end{pmatrix} , \quad (4.18)$$

⁶ Equation (4.17) also arises by demanding that (3.5) is satisfied for the background (4.1) and the complex structure (4.7) (in the coordinate basis (4.4) with $x^0 \equiv \psi$). Also we have omitted the obvious solution $R = 1$ that corresponds to the $SU(2) \otimes U(1)$ WZW model. Nevertheless, all of our formulae below that contain R explicitly will be valid for $R = 1$ as well. What is important to mention is that a marginal deformation away from the WZW point ($R = \text{const.} \neq 1$) leads to a loss of manifest $N = 4$, in agreement with the general statement of section 4 for WZW models based on quaternionic groups.

with the definitions

$$\begin{aligned}
f_1^\pm &= -d\rho \wedge d\varphi \pm \frac{R \tan \varphi}{1 + R^2 \tan^2 \varphi} d\tau \wedge d\tilde{\psi} \\
f_2^\pm &= \frac{1}{1 + R^2 \tan^2 \varphi} \left(-\tan \varphi d\rho \wedge (R^2 d\tau \mp d\tilde{\psi}) + R(d\tau \pm \tan^2 \varphi d\tilde{\psi}) \wedge d\varphi \right),
\end{aligned} \tag{4.19}$$

where the function R assumes either one of the two expressions in (4.17). Notice that, in agreement with what was expected for rotational-type Killing vectors [28,5] (see also appendix A), out of the three complex structures, \tilde{F}_3^\pm is a singlet of the duality group ($SO(2)$ in this case), whereas \tilde{F}_1^\pm and \tilde{F}_2^\pm form a doublet, in each chiral sector separately.

It is important to emphasize that in trying to obtain a 2-dim σ -model with manifest $N = 4$ supersymmetry from (4.1) via a duality transformation at no point we required that (4.1) or its dual was conformally invariant. The entire treatment was completely classical and the function $R(\rho)$ remained arbitrary. Both (4.1) and its dual (4.5) have non-locally realized $N = 4$ supersymmetry at the classical level. It turned out that the condition (4.17) that led to manifest $N = 4$ supersymmetry for the dual model is a particular case of (4.3), with $C_2 = -C_1 = 1$, that guarantee 1-loop conformal invariance for both models. For these choices for $R(\rho)$, (4.1)(4.2) correspond to the direct product $SU(2)/U(1)_k \otimes SL(2, \mathbb{R})_{-k-4}/U(1)$ and the central charge deficit is zero. Also $\Psi_{\pm,+}^{(2)}$ and $\bar{\Psi}_{\pm,-}^{(2)}$ become the usual classical non-compact parafermions of the $SL(2, \mathbb{R})/U(1)$ coset.

4.1. Spacetime Supersymmetry

In [7] a partial proof was given that the model (4.5) with the function $R(\rho)$ satisfying (4.17) has manifest spacetime supersymmetry by showing that only with these choices the dilatino equation can be satisfied. In view of possible subtleties [35] we complete the proof of [7] by solving the gravitino equation and finding the corresponding Killing spinors.

The Killing spinor equations are

$$\begin{aligned}
\delta\Psi_\mu &= \left(\partial_\mu + \frac{1}{4}(\omega_\mu^{\alpha\beta} - \frac{1}{2}H_\mu^{\alpha\beta})\gamma_{\alpha\beta} \right) \xi = 0, \\
\delta\lambda &= -\left(\gamma^\mu \partial_\mu \Phi + \frac{1}{6}H_{\mu\nu\lambda}\gamma^{\mu\nu\lambda} \right) \xi = 0,
\end{aligned} \tag{4.20}$$

where Ψ_μ and λ are the gravitino and dilatino fields respectively and the dilaton Φ is given by (4.2). We find that for the background (4.5) with the choice for $R = \tanh \rho$ or $R = \coth \rho$ the Killing spinor is

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = e^{-i(a_1\sigma_1 + a_3\sigma_3)(\tau + \psi)} e^{-ia_2\sigma_2} \begin{pmatrix} 0 \\ \epsilon_- \end{pmatrix}, \quad (4.21)$$

where ϵ_- is the non-zero Weyl component of a constant spinor and

$$\begin{aligned} a_1 &= \frac{1}{2} R \tan \varphi (1 + R^2 \tan^2 \varphi)^{-1/2}, & a_3 &= \frac{1}{2} (1 + R^2 \tan^2 \varphi)^{-1/2}, \\ a_2 &= \frac{1}{2} \tan^{-1}(R \tan \varphi). \end{aligned} \quad (4.22)$$

Notice that contrary to the case of restoration of manifest world-sheet supersymmetry which required no quantum input at all (conformal invariance was not even an issue), restoring manifest target space supersymmetry demanded the use of the dilaton Φ which is a 1-loop quantum effect in the α' -expansion.

4.2. Interpretation as a gauged WZW model

It is useful to associate the background (4.5) for the special cases where $N = 4$ extended world-sheet and spacetime supersymmetry become manifest, with a gauged WZW model. Consider the gauged WZW type action

$$S = k I_{wzw}(h_+^{-1} g_1 h_-) - k I_{wzw}(h_+^{-1} g_2 h_-), \quad (4.23)$$

for the group elements $g_1 \in SU(2)$ and $g_2 \in SL(2, \mathbb{R})$ parametrized as

$$g_1 = e^{i\sigma_1\theta_L} e^{i\sigma_3\varphi} e^{i\sigma_1\theta_R}, \quad g_2 = e^{i\sigma_1\omega_L} e^{\sigma_3\rho} e^{i\sigma_1\omega_R}, \quad (4.24)$$

and where h_\pm are two $U(1)$ group elements that parametrize the gauge fields $A_\pm \equiv h_\pm^{-1} \partial_\pm h_\pm$. Notice that the action (4.23) does not contain the typical for gauged WZW models term $I_{wzw}(h_+^{-1} h_-)$. Nevertheless, as we shall see bellow, the gauge field dependence of (4.23) is expressible in terms of gauge fields in a local way even without such a

term. The action (4.23) is manifestly invariant under the 2-parameter finite gauge transformation

$$\begin{aligned}\delta\theta_L &= \delta\omega_L = \epsilon_L , & \delta\varphi &= 0 , & h_+ &\rightarrow e^{i\sigma_1\epsilon_L} h_+ \\ \delta\theta_R &= \delta\omega_R = \epsilon_R , & \delta\rho &= 0 , & h_- &\rightarrow e^{-i\sigma_1\epsilon_R} h_- ,\end{aligned}\tag{4.25}$$

with gauge parameters $\epsilon_{L,R} = \epsilon_{L,R}(\sigma^+, \sigma^-)$. The gauge choice $\omega_L = \omega_R = 0$ completely fixes the gauge. Using the Polyakov–Wiegman formula and changing variables as $\theta_L = \frac{1}{2}(\tau - \psi)$ and $\theta_R = \frac{1}{2}(\tau + \psi)$ the action (4.23) takes the form

$$\begin{aligned}S &= \frac{k}{\pi} \int \partial_+ \varphi \partial_- \varphi + \partial_+ \rho \partial_- \rho + \cos^2 \varphi \partial_+ \tau \partial_- \tau + \sin^2 \varphi \partial_+ \psi \partial_- \psi \\ &\quad + \frac{1}{2} \cos 2\varphi (\partial_+ \tau \partial_- \psi - \partial_+ \psi \partial_- \tau) + 2iA_+ (\cos^2 \varphi \partial_- \tau - \sin^2 \varphi \partial_- \psi) \\ &\quad - 2i(\cos^2 \varphi \partial_+ \tau + \sin^2 \varphi \partial_+ \psi)A_- + 2A_+ A_- (\cos 2\varphi - \cosh 2\rho) .\end{aligned}\tag{4.26}$$

It is a standard straightforward procedure to integrate out the gauge fields and obtain a σ -model action (the non-trivial dilaton is also induced). It turns out that this model is equivalent to (4.5), with $R = \coth \rho$. Notice that in (4.26) as a consequence of (4.23) there is no $A_+ A_-$ -term with constant coefficient. This is a characteristic of what is known as “chiral” gauged WZW models [36,37]. We believe that the type of gauging (4.23) may lead to other models with manifest $N = 4$ supersymmetry.

4.3. The spacetime

In order to obtain a clear geometrical picture it is convenient to use Cartesian coordinates

$$\begin{aligned}x_1 &= r_0 \sinh \rho \cos \varphi \cos \tau , & x_2 &= r_0 \sinh \rho \cos \varphi \sin \tau , \\ x_3 &= r_0 \cosh \rho \sin \varphi \cos \tilde{\psi} , & x_4 &= r_0 \cosh \rho \sin \varphi \sin \tilde{\psi} ,\end{aligned}\tag{4.27}$$

if $R = \tanh \rho$ and similarly if $R = \coth \rho$, where r_0 is an arbitrary radial parameter. Then the background (4.5) takes the form

$$\begin{aligned}ds^2 &= e^{-\Phi} dx_i dx_i , & H_{ijk} &= -\epsilon_{ijk}{}^l \partial_l \Phi , \\ \Phi &= \frac{1}{2} \ln((x_i x_i + r_0^2)^2 - 4r_0^2(x_3^2 + x_4^2)) ,\end{aligned}\tag{4.28}$$

where we have omitted the tildes and for convenience we have presented the expression for the antisymmetric field strength, instead of the tensor itself. The metric is conformally flat with the conformal factor satisfying the Laplace equation adapted to the flat metric, i.e. $\partial_i \partial_i e^{-\Phi} = 0$, in agreement with a general theorem proved in [16].⁷ The antisymmetric field strength solves the (anti)self-duality conditions of the dilaton-axion field and therefore our solution (4.28) is an axionic-instanton. For completeness we write down the form of the complex structures in the coordinate system (4.27)

$$\begin{aligned} F_1^\pm &= e^{-\Phi} (-dx_1 \wedge dx_3 \pm dx_2 \wedge dx_4) , \\ F_2^\pm &= e^{-\Phi} (\pm dx_1 \wedge dx_4 + dx_2 \wedge dx_3) , \\ F_3^\pm &= e^{-\Phi} (dx_1 \wedge dx_2 \pm dx_3 \wedge dx_4) . \end{aligned} \tag{4.29}$$

In fact these are complex structures for all 4-dim axionic instantons of the form (4.28) irrespectively of the particular dilaton Φ .

Geometrically the metric in (4.28) represents a semi-wormhole with a fat throat.⁸ The metric has singularities not at a single point, but in the ring $x_1 = x_2 = 0$, $x_3^2 + x_4^2 = r_0^2$ (obviously the radius of the ring can be set equal to 1 by an overall rescaling of the coordinates). Therefore the throat never becomes infinitely thin. In the region around the origin at $x_i = 0$ (equivalently, if we let the ring radius become very large $r_0 \rightarrow \infty$) the background (4.28) becomes that corresponding to flat space with constant dilaton and antisymmetric tensor. Far away from the ring we expect that its presence should not play any role. Indeed, if we let $x_i = y_i/\epsilon$ and $\epsilon \rightarrow 0$ (equivalently if we let $r_0 \rightarrow 0$) we obtain a

⁷ This theorem states that in 4-dim backgrounds with $N = 4$ extended world-sheet supersymmetry and torsion, the metric is conformally related to a hyper-kahler one with the conformal factor satisfying the Laplace equation as defined using the hyper-kahler metric. This conclusion is not always true in cases with $(4, 0)$ world-sheet supersymmetry [17]. Nevertheless, it is true in our case where we have $(4, 4)$ world-sheet supersymmetry.

⁸ Actually it represents the throat of the wormhole itself. A true semi-wormhole is obtained only by shifting $e^{-\Phi}$ by a constant since then asymptotically the space is Euclidean. In our case this corresponds to an S -duality transformation. The same remarks hold for the $SU(2) \otimes U(1)$ semi-wormhole (see below).

solution of the form (4.28) but with $\Phi = \ln(y_i y_i)$. This is exactly the background of the usual semi-wormhole corresponding to the $SU(2) \otimes U(1)$ WZW model (with a background charge) [16,8]. Note that due to the ring singularity structure the isometry group of the usual semi-wormhole metric $O(4)$ breaks down to $SO(2) \otimes SO(2)$ for our semi-wormhole. Let us finally mention that close to the singularity ring the isometry group of the metric is enhanced to $SO(3)$. This can be seen by letting $x_i = 2\epsilon y_i$, for $i = 1, 2, 4$, $x_3 = r_0 + 2\epsilon y_3$ and then taking the limit $\epsilon \rightarrow 0$ and absorbing a factor ϵ into a redefinition of the string coupling α' (at the CFT level this corresponds to a contraction). The resulting axionic instanton is again of the form (4.28) but with $\Phi = \frac{1}{2} \ln(y_1^2 + y_2^2 + y_3^2)$, thus revealing the advertised $SO(3)$ isometry. It can be shown that this space is duality related to flat space.

The background (4.28) is the most general axionic instanton in 4-dim for which there is manifest $N = 4$ supersymmetry and the corresponding CFT is known. It encompasses every other similar solution that has appeared in the literature so far [8] since they are either duality related to it or they can be obtained from it via a combination of duality and contraction procedures of the type we have described above.

5. Concluding remarks

In this paper we provided general conditions for the existence of non-locally realized extended world-sheet supersymmetry in 2-dim supersymmetric σ -models. This has implications for the background fields. For instance, we prove in appendix A, among other theorems, that $N = 4$ (realized non-locally) does not imply that the manifold is Ricci flat in the absence of torsion. Next we examined the question of restoring manifest (equivalently locally realized) supersymmetry via duality transformations and gave general necessary conditions for being able to do that. Such restoration happens when due to non-local world-sheet effects taking place in the duality transformation, the non-local complex structures become local. This is the reverse mechanism of that destroying manifest supersymmetry. In the case that the underlying CFT is known the non-localities are

better represented by the usual parafermionic objects. This was explicitly demonstrated in a new 4-dim axionic instanton representing a semi-wormhole (generalizing the one corresponding to $SU(2) \otimes U(1)$) with a fat throat. The manifest spacetime supersymmetry of this background was also explicitly demonstrated by solving the Killing spinor equations.

The emergence of a manifest $N = 4$ raises some interesting questions concerning realizations of the $N = 4$ superconformal algebra. It even suggests the existence of new type of parafermions that are the fundamental symmetry generating objects in these backgrounds. Specifically, since our axionic instanton came as a particular duality transformation on the background corresponding to the tensor product $SU(2)_k/U(1) \otimes SL(2, \mathbb{R})_{-k-4}/U(1)$, the starting point in any realization of the $N = 4$ superconformal algebra would be to use the corresponding compact and non-compact parafermions [8]. However, as we have seen in each chiral sector one combination of these parafermions becomes a local object as a manifest $N = 4$ emerges. That suggests that only a combination of them, orthogonal to the first, behaves in a non-local manner, i.e., is a true parafermion. Since this was demonstrated only classically one should try to further develop this idea at the CFT level.

We believe that the general conditions for existence of non-local extended world-sheet supersymmetry we have presented in this paper should be used to explicitly demonstrate the hidden non-local supersymmetries of models obtained via non-abelian duality transformations [38]. Prototype examples are 4-dim $SO(3)$ -invariant hyper-kahler metrics. In these cases the non-abelian duality is performed with respect to the left (or right action) of the isometry group $SO(3)$. For a class of such metrics, that includes the Taub-NUT and the Atiyah-Hitchin, the three complex structures transform in the triplet representation of $SO(3)$ [28]. Non-abelian duality will break the original $N = 4$ as a local symmetry completely down to an $N = 1$ and a non-local realization will emerge [5]. It turns out that path ordered exponentials appear and the relevant equation to consider then is (2.7). Work along these lines is in progress.

It is also conceivable that duality and possibly dynamical restoration of manifest supersymmetry techniques would provide a natural explanation of a phenomenon observed in

[39] in $(2, 0)$ supersymmetric σ -models with one Killing symmetry and a complex structure non-preserved under diffeomorphisms. There, in order to close the symmetry algebra one had to “postulate” a compensating transformation for the complex structure. We are also convinced that restoration of manifest supersymmetry of the type we have exhibited in this paper can happen in more general Kazama-Suzuki models. We hope to report along these lines in the future [40].

We believe that the ideas and techniques developed in this paper could be used to explore the possibility that various solutions that are of interest in black hole physics or cosmology might have hidden supersymmetries or be related to solutions with manifest supersymmetry. Since this necessarily involves non-local world-sheet effects it would be important in our effort to understand the way string theory could resolve fundamental problems in physics.

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Appendix A. Theorems for non-locally realized $N = 4$

In this appendix we prove some basic theorems for backgrounds having non-local $N = 4$ extended supersymmetry of a type that can be described by the ansatz (2.5).

Suppose that we have three complex structures (non-local in general) obeying (2.4). Then each one of them satisfies (2.6) and its integrability condition (2.10) which we rewrite in a slightly different form after we multiply with a complex structure

$$R_{\mu\nu\alpha\beta}^{\pm} = R_{\mu\nu\gamma\delta}^{\pm} (F_I^{\pm})^{\gamma}_{\alpha} (F_I^{\pm})^{\delta}_{\beta} - 2\vec{C}_{\mu\nu}^{\mp} \cdot (\partial_{\vec{\theta}} F_I^{\pm} F_I^{\pm})_{\alpha\beta} , \quad (\text{A.1})$$

where I is being kept fixed and $\vec{C}_{\mu\nu}^{\pm} = -\frac{1}{2}\partial_{[\mu}\vec{C}_{\nu]}^{\pm}$ as in (2.11). Then we contract (A.1) by $(F_J^{\pm})^{\alpha\beta}$ to obtain

$$R_{\mu\nu\alpha\beta}^{\pm}(F_J^{\pm})^{\alpha\beta} = -R_{\mu\nu\alpha\beta}^{\pm}(F_I^{\pm}F_J^{\pm}F_I^{\pm})^{\alpha\beta} + 2\vec{C}_{\mu\nu}^{\mp} \cdot (\partial_{\vec{\theta}}F_I^{\pm}F_I^{\pm}F_J^{\pm})_{\alpha}^{\alpha} , \quad (\text{A.2})$$

with fixed I, J . After using (2.4) we finally obtain

$$R_{\mu\nu\alpha\beta}^{\pm}(F_J^{\pm})^{\alpha\beta} = R_{\mu\nu\alpha\beta}^{\pm}(F_I^{\pm})^{\alpha\beta}\delta^{IJ} + \vec{C}_{\mu\nu}^{\mp} \cdot (\partial_{\vec{\theta}}F_I^{\pm}F_K^{\pm})_{\alpha}^{\alpha}\epsilon^{IJK} , \quad (\text{A.3})$$

where we sum only over K . If $I = J$ this formula is trivial. Taking $I \neq J$ we obtain

$$R_{\mu\nu\alpha\beta}^{\pm}(F_J^{\pm})^{\alpha\beta} = \vec{C}_{\mu\nu}^{\mp} \cdot (\partial_{\vec{\theta}}F_I^{\pm}F_K^{\pm})_{\alpha}^{\alpha}\epsilon^{IJK} , \quad I \neq J . \quad (\text{A.4})$$

In the case of manifest $N = 4$ the right hand side is zero and we obtain (3.1).

Let us consider the torsionless case where the two generalized curvatures reduce to the Riemannian one. It is well established that then $N = 4$ supersymmetry implies Ricci flatness (see for instance [15]). We will see however that this is not the case for non-local $N = 4$. Using the cyclic identity $R_{\mu[\nu\alpha\beta]} = 0$ we obtain

$$R_{\mu\nu\alpha\beta}(F_I^{\pm})^{\nu\beta} = \frac{1}{2}R_{\mu\alpha\beta\nu}(F_I^{\pm})^{\beta\nu} . \quad (\text{A.5})$$

Then we contract (A.1) by $G^{\nu\beta}$ and use (A.5) to obtain

$$R_{\mu\nu} = -\frac{1}{2}R_{\mu\alpha\beta\gamma}(F_I^{\pm})^{\beta\gamma}(F_I^{\pm})_{\nu}^{\alpha} - 2\vec{C}_{\mu\alpha}^{\mp} \cdot (\partial_{\vec{\theta}}F_I^{\pm}F_I^{\pm})_{\nu}^{\alpha} . \quad (\text{A.6})$$

This could also be given a different form using (A.4). We see that generically only when $N = 4$ is local we obtain the usual Ricci flatness condition. Next we specialize to an important case where the above conditions can be simplified considerably.

A singlet and a doublet: Let us assume that there is only one θ . Having in mind the discussion of section 2 it can be easily seen that the only consistent possibility is to have two of the complex structures $F_{1,2}^{\pm}$ non-local and the third F_3^{\pm} local. Without loss of

generality we may choose $F_2^\pm = \pm \partial_\theta F_1^\pm$ and then from (2.4) it follows that $F_1^\pm = \mp \partial_\theta F_2^\pm$. Thus $F_{1,2}^\pm$ form a doublet in each sector separately and F_3^\pm is a singlet. Then a simple application of (A.4) gives

$$R_{\mu\nu\alpha\beta}^\pm (F_J^\pm)^{\alpha\beta} = \pm d C_{\mu\nu}^\mp \epsilon^{J12} , \quad (\text{A.7})$$

where d is the dimension of the target space. We see that only for the singlet, corresponding to the local complex structure, we get a non-zero result. For the doublet in contrast, we get zero. Consistency conditions for (A.7) are obtained after contracting it by $(F_3^\pm)^{\mu\nu}$ or by $(F_{1,2}^\pm)^{\mu\nu}$ and then using again (A.7) to reexpress the left hand side

$$(F_3^+)^{\mu\nu} C_{\mu\nu}^+ + (F_3^-)^{\mu\nu} C_{\mu\nu}^- = 0 , \quad (F_{1,2}^\pm)^{\mu\nu} C_{\mu\nu}^\pm = 0 . \quad (\text{A.8})$$

Before we turn to the torsionless case let us consider the case when the non-locality arises from an abelian duality transformation with respect to a Killing vector and, as usual, let us denote quantities in the dual model with tildes. Then, as we have mentioned, $\tilde{C}_\mu^\pm = \pm \tilde{Q}_{0\mu}^\mp$ and hence $\tilde{C}_{\mu\nu}^\pm = \mp \frac{1}{2} \partial_{[\mu} \tilde{Q}_{0\nu]}^\mp$. Using the duality transformation rules for background fields [1] we derive that

$$\tilde{C}_{\mu\nu}^\pm = (A_\pm^T C^\pm A_\pm)_{\mu\nu} , \quad (\text{A.9})$$

where $C_{\mu\nu}^\pm = \frac{1}{2} G_{00}^{-1} \partial_{[\mu} Q_{0\nu]}^\mp$. Then (A.4), (written in the dual model with tildes) can be rewritten in terms of tensors defined in the original model, with the help of (3.2)(3.4)(A.9).

We obtain

$$R_{\alpha\beta\mu\nu}^\pm (F_J^\pm)^{\mu\nu} + G_{00}^{-1} \partial_{[\alpha} Q_{0\beta]}^\pm \partial_\mu Q_{0\nu}^\mp (F_J^\pm)^{\mu\nu} = \frac{1}{2} G_{00}^{-1} \partial_{[\alpha} Q_{0\beta]}^\pm (\partial_\theta F_I^\pm F_K^\pm)_\gamma \epsilon^{IJK} , \quad (\text{A.10})$$

where $I \neq J$. Assuming that the original model has manifest $N = 4$ implies that the first term in the above equation is zero. If further the isometry is not chiral, i.e., $\partial_\alpha Q_{0\beta}^\pm \neq 0$, and using the doublet structure of the two non-local complex structures mentioned above we deduce that

$$\partial_\mu Q_{0\nu}^\mp (F_J^\pm)^{\mu\nu} = \pm \frac{d}{2} \epsilon^{J12} , \quad (\text{A.11})$$

which notably, is only a dimension dependent constant. Notice also the consistency with (A.8) and that the left hand side of (A.11) appears in (3.5).

If the torsion is zero then applying (A.6) for $I = 1$ (or $I = 2$) and using (A.7) gives

$$R_{\mu\nu} = \mp C_{(\mu\alpha}^{\mp} (F_3)^{\alpha}_{\nu)} . \quad (\text{A.12})$$

However, applying it for $I = 3$ we obtain the same formula only if the target space dimension is $d = 4$. Therefore we have proved that the ansatz that we can have non-local $N = 4$ in torsionless backgrounds with one complex structure being local and the other two forming a doublet is consistent only in $d = 4$ dimensions. In addition since the Ricci tensor $R_{\mu\nu}$ is the same in both chiral sectors we conclude that we should have $C_{\mu\nu}^- = -C_{\mu\nu}^+$ and therefore we may choose $C_{\mu}^- = -C_{\mu}^+$.

Using the explicit results of [5], we have verified the general structure we have exhibited here for hyper-kähler manifolds with a rotational Killing symmetry and their duals. The same checking was also done using the results of section 4.

Lets us conclude this appendix by proving that we cannot have three non-local complex structures which depend on three functionals θ^I and also transform as a triplet of $SO(3)$ in the sense that

$$\partial_{\theta^I} F_J^{\pm} = \epsilon^{IJK} F_K^{\pm} . \quad (\text{A.13})$$

After some algebra we find that (A.4) takes the simplified form

$$R_{\mu\nu\alpha\beta}^{\pm} (F_J^{\pm})^{\alpha\beta} = d C_{\mu\nu}^{\mp J} . \quad (\text{A.14})$$

However, since $C_{\mu\nu}^{\mp J}$ is independent of θ^I we can take the derivative of both sides of (A.14) with respect to θ^I , use (A.13) to rewrite the left hand side and finally obtain that $R_{\mu\nu\alpha\beta}^{\pm} (F_J^{\pm})^{\alpha\beta} = 0$, which when compared to (A.14) implies that $C_{\mu\nu}^{\pm J} = 0$ for all three values of J . Therefore $C_{\mu}^{\pm J}$ is a total derivative and θ^J is a local function.

Appendix B. “Dynamical” moduli and duality

In this appendix we prove, for a class of models, that making moduli parameters “dynamical” (equivalently coordinate dependent) and retaining conformal invariance is equivalent to performing duality transformations.

We start with an action corresponding to the tensor product $G/U(1) \otimes (2D)_R$, where the second factor denotes any of the 2-dim conformal models corresponding to the function $R(\rho)$ introduced in section 4. Using the definitions (3.8) we have

$$S = I_0(h) + \frac{1}{2\pi} \int E \partial_+ \alpha \partial_- \alpha + \frac{2}{1-\Sigma} (\partial_+ \alpha J_i^- \partial_- x^i + J_i^+ \partial_+ x^i \partial_- \alpha + J_i^+ J_j^- \partial_+ x^i \partial_- x^j) + \frac{1}{2\pi} \int \partial_+ \rho \partial_- \rho + R^2(\rho) \partial_+ \beta \partial_- \beta, \quad (\text{B.1})$$

where $E \equiv \frac{1+\Sigma}{1-\Sigma}$. After we let $\alpha = \tau - \psi/2$ and $\beta = \tau + \psi/2$ and perform a duality transformation with respect to the Killing vector $\partial/\partial\tau$ we obtain the action

$$\tilde{S} = I_0(h) + \frac{1}{2\pi} \int \partial_+ \rho \partial_- \rho + \frac{1}{E + R^2} \left(\partial_+ \tau \partial_- \tau + E(R^2 \partial_+ \psi \partial_- \psi + \partial_+ \psi \partial_- \tau - \partial_+ \tau \partial_- \psi) - 2 \frac{1-R^2}{1-\Sigma} J_i^+ J_j^- \partial_+ x^i \partial_- x^j + \frac{2}{1-\Sigma} ((\partial_+ \tau - R^2 \partial_+ \psi) J_i^- \partial_- x^i - J_i^+ \partial_+ x^i (\partial_- \tau + R^2 \partial_- \psi)) \right). \quad (\text{B.2})$$

We would like to compare this action with the one that follows from the WZW model for a group G marginally deformed by a current–bilinear in the Cartan torus. This is equivalent to the model dual to (3.7) and the deformation parameter is the modulus Q . The idea is then, to make the modulus a function of the target space variables [22,20,21]. In our case we add to the dual to (3.7) the term $\frac{1}{2\pi} \int \partial_+ \rho \partial_- \rho$ and for convenience we let $\psi \rightarrow \frac{1}{2}(Q\psi + \tau)$. The result becomes just (B.2) if we replace $Q \rightarrow R(\rho)$. Of course without trying to satisfy the β -function equations we do not know a priori the function $R(\rho)$. But the relation to the direct product $G/U(1) \otimes (2D)_R$ by a duality transformation tells us that conformal invariance constrains $R(\rho)$ to be given by (4.3).

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